ELSEVIER

Discrete Mathematics 197/198 (1999) 503–513

DISCRETE MATHEMATICS

Polarities and $2k$-cycle-free graphs [1]

Felix Lazebnik [a,*], Vasiliy A. Ustimenko [b], Andrew J. Woldar [c]

[a] *Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA*
[b] *Department of Mathematics and Mechanics, University of Kiev, Kiev 252127, Ukraine*
[c] *Department of Mathematical Sciences, Villanova University, Villanova, PA 19085, USA*

Received 9 July 1997; revised 18 December 1997; accepted 3 August 1998

Abstract

Let C_{2k} be the cycle on $2k$ vertices, and let $\mathrm{ex}(v, C_{2k})$ denote the greatest number of edges in a simple graph on v vertices which contains no subgraph isomorphic to C_{2k}. In this paper we discuss a method which allows one to sometimes improve numerical constants in lower bounds for $\mathrm{ex}(v, C_{2k})$. The method utilizes polarities in certain rank two geometries. It is applied to refute some conjectures about the values of $\mathrm{ex}(v, C_{2k})$, and to construct some new examples of graphs having certain restrictions on the lengths of their cycles. In particular, we construct an infinite family $\{G_i\}$ of C_6-free graphs with $|E(G_i)| \sim \frac{1}{2}|V(G_i)|^{4/3}$, $i \to \infty$, which improves the constant in the previous best lower bound on $\mathrm{ex}(v, C_6)$ from $2/3^{4/3} \approx 0.462$ to $1/2$.

1. Introduction

Let Γ be a simple graph (undirected, no multiple edges, no loops) and let $\mathscr{F}$ be a family of graphs none of which is isomorphic to a subgraph of Γ. In this case we say that Γ is *$\mathscr{F}$-free*. By $\mathrm{ex}(v, \mathscr{F})$ we denote the greatest number of edges of an $\mathscr{F}$-free graph on v vertices. If $\mathscr{F} = \{F\}$, we write $\mathrm{ex}(v, F)$ in place of $\mathrm{ex}(v, \{F\})$. C_n shall always denote the cycle on n vertices, $n \geqslant 3$. By $v = v(\Gamma) = |V(\Gamma)|$ and $e = e(\Gamma) = |E(\Gamma)|$ we denote the number of vertices and edges of Γ, respectively. If Γ contains cycles, then the length of a smallest cycle of Γ is called the *girth* of Γ and is denoted $g(\Gamma)$. Let f and g be two real valued functions on (a, ∞). To compare their behaviour for large x we use the following standard notations:
$f(x) \sim g(x)$, $x \to \infty$ if $f(x)/g(x) \to 1$ for $x \to \infty$;

[1] This research was supported by NSF grants DMS-9115473 and DMS-9622091. A.J. Woldar was additionally supported by a Villanova University Faculty Research Grant. V.A. Ustimenko was additionally supported in part by INTAS-93-2530 and INTAS-94-3420. F. Lazebnik was additionally supported by a University of Delaware Research Award.

* Corresponding author. E-mail: fellaz@math.udel.edu.

0012-365X/99/$ – see front matter
PII: S0012-365X(98)00258-1

$f(x)=\mathrm{o}(g(x))$, $x\to\infty$ if $f(x)/g(x)\to 0$ for $x\to\infty$;
$f(x)=\mathrm{O}(g(x))$, $x\to\infty$ if there exist C, x_0 such that $|f(x)|<Cg(x)$ for all $x>x_0$;
$f(x)=\Omega(g(x))$, $x\to\infty$ if there exists a $c>0$ and a sequence $x_1,x_2,\ldots\to\infty$ such that $|f(x_i)|>cg(x_i)$ for all $i\geqslant 1$.

If $\mathscr{F}$ contains no bipartite graphs, the asymptotics of $\mathrm{ex}(v,\mathscr{F})$, $v\to\infty$, and the structure of the corresponding extremal graphs, are well understood (see, e.g., [2,10] and the references therein). If, however, $\mathscr{F}$ contains a bipartite graph, (for example, an even cycle) then much less is known in the general case (see [10,25]).

Let $k\geqslant 2$ be a fixed but arbitrary integer. The following unpublished result of Paul Erdős is often referred to as The Even Circuit Theorem (see [25]):

$$\mathrm{ex}(v,C_{2k})=\mathrm{O}(v^{1+\frac{1}{k}}).$$

For a proof of this result and its generalization, see [3,12].

A general lower bound for $\mathrm{ex}(v,C_{2k})$ is also available. It follows from the obvious inequality $\mathrm{ex}(v,C_{2k})\geqslant\mathrm{ex}(v,\{C_3,C_4,\ldots,C_{2k+1}\})$ and a lower bound for the latter established by the authors in [19]:

$$\mathrm{ex}(v,\{C_3,C_4,\ldots,C_{2k+1}\})=\Omega(v^{1+\frac{2}{3k-3+\varepsilon}}), \tag{1.1}$$

where $\varepsilon=0$ if k is odd, and $\varepsilon=1$ if k is even. For $k=5$, a better lower bound $\Omega(v^{1+1/5})$ is achieved, e.g., by a regular generalized hexagon (see Section 3).

It is known that $\mathrm{ex}(v,C_{2k})=\Theta(v^{1+\frac{1}{k}})$ for $k=2,3,5$ (see [5,9] for $k=2$ and [1,16,28] for $k=3,5$). For other values of k the existence of a $2k$-cycle-free graph with $\Omega(v^{1+\frac{1}{k}})$ edges has not been established. The best results are obtained for $k=2$, where the asymptotics $\mathrm{ex}(v,C_4)\sim\frac{1}{2}v^{3/2}$ were established in [5,9]. The exact values of $\mathrm{ex}(v,C_4)$ are known for all $1\leqslant v\leqslant 21$, see [7]. They are also known for all v of the form $v=q^2+q+1$, where q is either a power of 2 [13] or a prime power exceeding 13 [14]; indeed, $\mathrm{ex}(v,C_4)=\frac{1}{2}q(q+1)^2$ in these cases.

In [10] the upper bound

$$\mathrm{ex}(v,\{C_3,C_4,\ldots,C_{2k+1}\})\leqslant(\tfrac{1}{2})^{1+\frac{1}{k}}v^{1+\frac{1}{k}}+2^k(v/2)^{1-\frac{1}{k}}$$

was established for all integers $k\geqslant 1$, and it was conjectured, for all integers k, $p\geqslant 2$, that

$$\mathrm{ex}(v,\{C_{2k},C_{2p-1}\})=(\tfrac{1}{2})^{1+\frac{1}{k}}v^{1+\frac{1}{k}}+\mathrm{o}(v^{1+\frac{1}{k}}), \tag{1.2}$$

$$\mathrm{ex}(v,C_{2k})=\tfrac{1}{2}v^{1+\frac{1}{k}}+\mathrm{o}(v^{1+\frac{1}{k}}). \tag{1.3}$$

In our earlier paper [18] these conjectures were disproved for certain values of k, but we were unaware of this at the time the paper was written. Conjecture (1.2) was disproved for $k=3,5$ and all $p\geqslant 2$ via a construction of $2k$-cycle-free bipartite graphs of order v containing

$$\frac{k-1}{k^{1+\frac{1}{k}}} v^{1+\frac{1}{k}} + o(v^{1+\frac{1}{k}})$$

edges. For $k=5$, the same graphs refuted (1.3), since $4/5^{6/5} \approx 0.58 > 1/2$. Therefore the current status of (1.3) is that it is correct for $k=2$ and false for $k=5$.

In this paper we discuss another method which again allows one to sometimes improve numerical constants in lower bounds for $\mathrm{ex}(v, C_{2k})$. Immediate applications of this method allow us to construct infinite families of non-bipartite graphs which meet the conjectured upper bound (1.3) for $k=3$ and 5. For $k=3$, the obtained lower bound improves the constant in the previously best known lower bound [18] from $2/3^{4/3} \approx 0.462$ to $1/2$. To our knowledge, this is best known. As we have already mentioned, for $k=5$ the constant $1/2$ is lower than the one obtained in [18]. Also the graphs introduced in this paper are the first examples of non-bipartite graphs which refute (1.2) for $(k,p)=(3,2),(5,2),(5,3)$.

Our method is motivated by, and is a generalization of, an idea from [5,9] of using a polarity of the projective plane $PG(2,q)$ to show that $\mathrm{ex}(v, C_4) = \frac{1}{2}v^{3/2} + o(v^{3/2})$. Polarities of geometries and their properties are discussed in Section 2. On the other hand, finite projective planes are exactly the thick regular generalized triangles (regular generalized polygons are discussed in Section 3).Therefore it is natural to investigate polarities in other generalized polygons or rank two geometries (defined in Section 2) in an attempt to construct graphs which improve best known lower bounds for $\mathrm{ex}(v, C_{2k})$, $k \geqslant 3$. The problem with this approach is that (i) there are very few classes of generalized polygons, (ii) not all generalized polygons have 'enough' edges, and (iii) not all generalized polygons possess polarities. Nevertheless, in Section 3 we use polarities in certain generalized quadrangles and hexagons to obtain the best known constructions mentioned above.

In Section 4 we describe the graphs $\mathrm{CD}(n,q)$ from [19] which provide the lower bound in (1.1), and we exhibit polarities in such graphs when n is even and $q=2^\alpha$. These graphs are of interest because they are obtained from rank two geometries which possess polarities but are not generalized polygons. Though the polarities enable us to improve the constants in the asymptotic lower bound for $\mathrm{ex}(v, C_{2k})$ when $k=4$ and $k \geqslant 7$, these improvements are only marginally interesting since (i) better constants are provided by the method of [18], and (ii) it is not known whether these graphs are magnitude extremal for $k=4$ and $k \geqslant 7$.

2. Polarities of geometries

Let $\mathscr{P}$ and $\mathscr{L}$ be disjoint sets, the elements of which we call *points* and *lines*, respectively. A subset $I \subseteq \mathscr{P} \times \mathscr{L}$ is called an *incidence relation* on the pair $(\mathscr{P}, \mathscr{L})$, and the triple $(\mathscr{P}, \mathscr{L}, I)$ is called a *(rank two) geometry*. The *incidence graph* Γ of geometry $(\mathscr{P}, \mathscr{L}, I)$ is defined to be the bipartite graph with vertex set $\mathscr{P} \cup \mathscr{L}$ and edge set $\{pl \mid p \in \mathscr{P}, l \in \mathscr{L}, (p,l) \in I\}$.

Let $\pi : \mathscr{P} \cup \mathscr{L} \to \mathscr{P} \cup \mathscr{L}$ be a bijection for which the following hold:

(i) $\mathscr{P}^\pi = \mathscr{L}$ and $\mathscr{L}^\pi = \mathscr{P}$,

(ii) $\forall p \in \mathscr{P}, \forall l \in \mathscr{L}, \quad (l^\pi, p^\pi) \in I \Leftrightarrow (p,l) \in I$,

(iii) $\pi^2 = 1$.

We call such π a *polarity* of the geometry $(\mathscr{P}, \mathscr{L}, I)$. Note that π induces an order two automorphism of the incidence graph Γ which interchanges the bipartition sets $\mathscr{P}$ and $\mathscr{L}$. We shall use the term 'polarity' and the notation 'π' for this automorphism as well.

We now define the *polarity graph* Γ^π of geometry $(\mathscr{P}, \mathscr{L}, I)$ with respect to polarity π. It is the graph with vertex set $V(\Gamma^\pi) = \mathscr{P}$ and edge set $E(\Gamma^\pi) = \{p_1 p_2 \mid p_1, p_2 \in \mathscr{P}, p_1 \neq p_2, \ (p_1, p_2^\pi) \in I\}$.

Finally, we call point $p \in \mathscr{P}$ an *absolute point* of the polarity π provided $(p, p^\pi) \in I$. Let N_π denote the number of absolute points of π.

Theorem 1. *Let π be a polarity of the geometry $(\mathscr{P}, \mathscr{L}, I)$, and let Γ and Γ^π be the corresponding incidence and polarity graphs.*

(i) *$deg_{\Gamma^\pi}(p) = deg_\Gamma(p) - 1$ if p is an absolute point of π, and $deg_{\Gamma^\pi}(p) = deg_\Gamma(p)$ otherwise.*

(ii) $|V(\Gamma^\pi)| = \frac{1}{2}|V(\Gamma)|, \ |E(\Gamma^\pi)| = |E(\Gamma)| - N_\pi$.

(iii) *If Γ^π contains a $(2k+1)$-cycle then Γ contains a $(4k+2)$-cycle.*

(iv) *If Γ^π contains a $2k$-cycle then Γ contains two vertex disjoint $2k$-cycles C and C' such that $C^\pi = C'$. Consequently, if Γ is $2k$-cycle-free then so is Γ^π.*

(v) *The girths of the two graphs are related by $g(\Gamma^\pi) \geqslant \frac{1}{2} g(\Gamma)$.*

Proof. Parts (i) and (ii) are obvious. For (iii), let $p_1, p_2, \ldots, p_{2k+1}$ be consecutive vertices of a $(2k+1)$-cycle in Γ^π. Then

$$p_1, (p_2)^\pi, p_3, (p_4)^\pi, \ldots, p_{2k+1}, (p_1)^\pi, p_2, (p_3)^\pi, \ldots, p_{2k}, (p_{2k+1})^\pi$$

are consecutive vertices of a $(4k+2)$-cycle in Γ.

For (iv), let $p_1, p_2, \ldots, p_{2k}$ be consecutive vertices of a $2k$-cycle in Γ^π. Then each of

$$p_1, (p_2)^\pi, p_3, (p_4)^\pi, \ldots, p_{2k-1}, (p_{2k})^\pi,$$

$$(p_1)^\pi, p_2, (p_3)^\pi, p_4, \ldots, (p_{2k-1})^\pi, p_{2k}$$

is a string of consecutive vertices of a $2k$-cycle in Γ. Naming these $2k$-cycles C and C', respectively, it is trivial to see that C and C' are vertex disjoint and $C^\pi = C'$.

The proof of (v) follows immediately from (iii) and (iv). $\square$

3. Polarities of generalized polygons

We define a *regular generalized m-gon* as a rank two geometry $(\mathscr{P},\mathscr{L},I)$ whose incidence graph Γ is r-regular of girth $2m$ and diameter m. (We caution the reader that our use of the term 'regular' in this definition differs from its more conventional use in the context of generalized polygons.) For $r=2$ and all $m\geqslant 3$ a generalized m-gon always exists: it is the geometry $(\mathscr{P},\mathscr{L},I)$ where $\mathscr{P}=V(C_m)$, $\mathscr{L}=E(C_m)$, and I is the usual incidence between vertices and edges of the m-cycle C_m. (Note that $\Gamma=C_{2m}$ in this case.)

If $r\geqslant 3$, a generalized m-gon is called *thick* and its existence is quite rare. In [11] Feit and Higman proved that finite thick regular generalized m-gons exist only for $m=3,4$, and 6. We refer to these as regular generalized triangles, quadrangles and hexagons, respectively.

The notion of generalized polygons was introduced by Tits in [23], and examples of them can be constructed via the following scheme. Let G be a group and P_1,P_2 be two distinct subgroups of G. Let $\mathscr{P}=(G:P_1)$ and $\mathscr{L}=(G:P_2)$ be the sets of left cosets of P_1 and P_2 in G, respectively, and let incidence be described by $I=\{(\alpha,\beta)\mid\alpha\in\mathscr{P},\beta\in\mathscr{L},\alpha\cap\beta\neq\emptyset\}$. For certain choices of G,P_1,P_2, the obtained geometry turns out to be a regular generalized m-gon. For example, when $G\cong A_2(q),B_2(q),G_2(q),q=p^a$ a prime power (the so-called rank two Chevalley groups of normal type [6]), we obtain a $(q+1)$-regular generalized m-gon for $m=3,4,6$, respectively. In all these cases P_1 and P_2 are maximal subgroups of G containing the normalizer B of a fixed Sylow p-subgroup of G. (B is called a Borel subgroup of G and the P_i's are called maximal parabolics of G.) For these generalized m-gons, $|\mathscr{P}|=|\mathscr{L}|=1+q+q^2+\cdots+q^{m-1}$, see [22] for details.

Clearly B acts on each of $\mathscr{P},\mathscr{L}$ of the m-gon $(\mathscr{P},\mathscr{L},I)$ by left multiplication. It is known [6] that the orbits of B on each of these sets have lengths $1,q,q^2,\ldots,q^{m-1}$. Let $\mathscr{P}_*$ and $\mathscr{L}_*$ denote the largest orbits of B on $\mathscr{P}$ and $\mathscr{L}$, respectively (so $|\mathscr{P}_*|=|\mathscr{L}_*|=q^{m-1}$). Then the geometry $(\mathscr{P}_*,\mathscr{L}_*,I_*)$, where I_* denotes the restriction of I to $\mathscr{P}_*\times\mathscr{L}_*$, is called the *affine subgeometry*, or *affine part*, of $(\mathscr{P},\mathscr{L},I)$. The incidence graph Γ_* of $(\mathscr{P}_*,\mathscr{L}_*,I_*)$ is the subgraph of Γ induced on the set $\mathscr{P}_*\cup\mathscr{L}_*$.

If the generalized m-gon $(\mathscr{P},\mathscr{L},I)$ admits a polarity, then one can be chosen — call it π — for which $\mathscr{P}_*\cup\mathscr{L}_*$ is setwise stable. Denote by π_* the restriction of π to the subgeometry $(\mathscr{P}_*,\mathscr{L}_*,I_*)$. It is well known that for regular generalized triangles of type $A_2(q)$, a polarity π (so also π_*) exists for every prime power q. For regular generalized 4-gons of type $B_2(q)$, a polarity exists if and only if $q=2^{2\alpha+1}$, and for 6-gons of type $G_2(q)$, a polarity exists if and only if $q=3^{2\alpha+1}$ (e.g., see [24] or [6]).

Remark. The existence of these polarities can be explained as follows. Each Coxeter diagram of the type A_2,B_2 and G_2 admits an order two symmetry ϕ. In the case of A_2,ϕ extends to an involutory outer automorphism ϕ' of the group $A_2(q)$ for all fields $GF(q)$, and ϕ' induces a well known contragradient automorphism of the Desarguesian projective plane. In contrast, the symmetries of B_2 and G_2 diagrams can be extended to the corresponding group automorphism ϕ' only for special values of q. This automorphism ϕ' is a so-called diagram automorphism of the group (see [6] for more details). Another class of symmetries contains 'field automorphisms of the group', which are extentions of elements of $\mathrm{Aut}(GF(q))$ to group automorphisms. There is an involutary composition of the diagram and field automorphisms which induces a polarity of the geometry.

The following statement follows immediately from the results of Sections 4 and 5 of [24]; see also the description of absolute points of π_* in the affine models at the end of this section.

Proposition 2. *Let $(\mathscr{P},\mathscr{L},I)$ be a generalized m-gon of type $A_2(q), B_2(2^{2\alpha+1})$ or $G_2(3^{2\alpha+1})$, and let π be the polarity of $(\mathscr{P},\mathscr{L},I)$ described in the remark. Then π has precisely $q^{\lfloor \frac{m}{2} \rfloor}+1$ absolute points, $q^{\lfloor \frac{m}{2} \rfloor}$ of which lie in $\mathscr{P}_*$.*

Combinations of parts of Theorem 1 and Proposition 2 allow us to determine those parameters of graphs $\Gamma, \Gamma^{\pi}, \Gamma_*$, and Γ_*^{π} which are pertinent to this paper. We collect this information below.

Incidence graph Γ:
- $|\mathscr{P}|=|\mathscr{L}|=q^{m-1}+\cdots+q+1$, $v(\Gamma)=2(q^{m-1}+\cdots+q+1)$.
- Γ is $(q+1)$-regular, $e(\Gamma)=(q+1)(q^{m-1}+\cdots+q+1)$.
- $e\sim(\frac{1}{2})^{m/(m-1)}v^{m/(m-1)}$.
- $g(\Gamma)=2m$, $\mathrm{diam}(\Gamma)=m$.

Polarity graph Γ^{π}:
- $v(\Gamma^{\pi})=q^{m-1}+\cdots+q+1$.
- $2e(\Gamma^{\pi})=e(\Gamma)-N_{\pi}(\Gamma)=(q+1)(q^{m-1}+\cdots+q+1)-(q^{\lfloor \frac{m}{2} \rfloor}+1)$.
- $e\sim\frac{1}{2}v^{m/(m-1)}$.
- Γ^{π} is $(2m-2)$-cycle-free.

Remark. In the case $m=3$ (generalized triangle of type $A_2(q)$), the geometry $(\mathscr{P},\mathscr{L}, I)$ provides a familiar model of the projective plane, wherein P_1, P_2 and B correspond to the stabilizers in G of a point, line and flag, respectively. Moreover, in this case graph Γ^{π} is isomorphic to the graph from [5,9].

Affine graph Γ_*:
- $|\mathscr{P}_*|=|\mathscr{L}_*|=q^{m-1}$, $v(\Gamma_*)=2q^{m-1}$.
- Γ_* is q-regular, $e(\Gamma_*)=q^m$.
- $e\sim(\frac{1}{2})^{m/(m-1)}v^{m/(m-1)}$.
- $g(\Gamma_*)\geqslant 2m$, $\mathrm{diam}(\Gamma_*)>m$.

Remark. It is shown in [4] that for q sufficiently large, $\mathrm{diam}(\Gamma_*)=m+1$. It is also shown that for certain small $q, g(\Gamma_*)>2m$ and/or $\mathrm{diam}(\Gamma_*)=\infty$.

Polarity graph Γ_*^π:

- $v(\Gamma_*^\pi) = q^{m-1}$.
- $2e(\Gamma_*^\pi) = e(\Gamma_*) - N_\pi(\Gamma_*) = q^m - q^{\lfloor \frac{m}{2} \rfloor}$.
- $e \sim \frac{1}{2} v^{m/(m-1)}$.
- Γ_*^π is $(2m-2)$-cycle-free.

Remark. Concentrating on the asymptotics of e (number of edges) as a function of v (number of vertices), graphs Γ_*^π do not lead to a better numerical constant than the one provided by graphs Γ^π. They do however have greater edge density (i.e. greater ratio $e/\binom{v}{2}$) than do the latter graphs. This statement is also true if we replace Γ_*^π and Γ^π by Γ_* and Γ, respectively.

Below we provide coordinatized models of the affine parts of the generalized triangles and quadrangles, as well as descriptions of their corresponding polarities. In doing this our goal is two-fold. First, such representations are useful in describing the associated graphs in a group-free setting, thus allowing alternative methods for establishing properties of the graphs. (In some cases these methods seemed preferable to the group-theoretic ones.) Second, such representations often point us toward interesting generalizations, see Section 4 for an example of this. The possibility of such coordinatizations for all geometries of simple groups of Lie type in the case of fields of sufficiently large characteristic follows from [26] (see also [27]). The following models are merely restrictions of these coordinatizations in the case of the $A_2(q), B_2(q)$ geometries. These affine models are valid for arbitrary characteristic. (A nice alternate coordinatization of classical generalized polygons and their polarities can be found in [8,15].)

Affine model for generalized triangle of type $A_2(q)$: It is possible to show that over an arbitrary finite field $GF(q)$, the affine part of the generalized triangle is isomorphic to the following bipartite graph:

$$\mathscr{P}_* = \{(a,b) \mid a,b \in GF(q)\}, \quad \mathscr{L}_* = \{[c,d] \mid c,d \in GF(q)\}$$

$$((a,b),[c,d]) \in I_* \Leftrightarrow d - b = ca.$$

and the polarity π_* can be described by

$$\pi_* : (a,b) \leftrightarrow [a,-b].$$

Hence its corresponding absolute points have the form

$$(a, -\tfrac{1}{2}a^2), \quad a \in GF(q), \text{when } q \text{ is odd, and}$$

$$(0,a), \quad a \in GF(q), \text{when } q \text{ is even.}$$

In any case, π_* has precisely q absolute points, which yields the parameters of the polarity graphs Γ_*^π of Γ_* presented earlier in this section.

Affine model for generalized quadrangle of type $B_2(q)$: It can be shown (using [16] for q odd and [21] for q even) that the affine part of the generalized quadrangle is isomorphic to the following bipartite graph:

$$\mathscr{P}_* = \{(a,b,c) \mid a,b,c \in GF(q)\}, \quad \mathscr{L}_* = \{[d,e,f] \mid d,e,f \in GF(q)\}$$

$$((a,b,c),[d,e,f]) \in I_* \Leftrightarrow e-b=da, f-c=ea.$$

It is known [21] that when $q=2^{2\alpha+1}$, the polarity π_* can be described by

$$\pi_* : (a,b,c) \mapsto [a^{2^{\alpha+1}}, (ab)^{2^\alpha} + c^{2^\alpha}, b^{2^{\alpha+1}}],$$

$$\pi_* : [d,e,f] \mapsto (d^{2^\alpha}, f^{2^\alpha}, e^{2^{\alpha+1}} + (df)^{2^\alpha}).$$

In this case one can easily check that its corresponding absolute points have the form

$$(a,b,a^{2^{\alpha+1}+2} + ab + b^{2^{\alpha+1}}), \quad a,b \in GF(q).$$

Thus π_* has precisely q^2 absolute points and this yields the parameters of the polarity graphs Γ_*^π of Γ_* presented earlier in this section.

Remarks. 1. The fact that the graphs described above are isomorphic to the affine parts of the generalized triangle and the generalized quadrangle are not essential in this paper, which focuses on the extremal properties. What is important is that their order, size and girth are the same as in the affine parts of the corresponding generalized polygons, and that they possess polarities. All these facts can be easily verified from the definitions of the graphs and their polarities in terms of coordinates which were presented above.

2. We present here neither a coordinatized model for the affine part of the generalized hexagon of type $G_2(q)$, nor a description of the corresponding polarity when $q=3^{2\alpha+1}$. The main reason is that the numerical constant achieved by the associated polarity graphs, though an improvement over what is obtained from the incidence graphs, is not best known. As mentioned in the Introduction, the best known constant $4/5^{6/5}$ is achieved as the result of a different technique of the authors, see [18]. We only remark once more that such a polarity exists when $q=3^{2\alpha+1}$ and that q^3 of its q^3+1 absolute points lie in its affine subgeometry.

4. Polarities of graphs CD(n, q)

We begin by considering the graph $D(n,q)$ introduced in [17]. Let q be any prime power, and let $\mathscr{P}$ and $\mathscr{L}$ be two copies of the countably infinite dimensional vector space V over $GF(q)$. As usual, elements of $\mathscr{P}$ will be called points and those of $\mathscr{L}$ lines. In order to distinguish points from lines we introduce the use of parentheses and brackets: If $v \in V$, then $(v) \in \mathcal{P}$ and $[v] \in \mathcal{L}$. We adopt the notation for coordinates of points and lines used in [17]:

$$(p) = (p_1, p_{11}, p_{12}, p_{21}, p_{22}, p'_{22}, p_{23}, p_{32}, \ldots, p_{ii}, p'_{ii}, p_{i,i+1}, p_{i+1,i}, \ldots),$$
$$[l] = [l_1, l_{11}, l_{12}, l_{21}, l_{22}, l'_{22}, l_{23}, l_{32}, \ldots, l_{ii}, l'_{ii}, l_{i,i+1}, l_{i+1,i}, \ldots].$$

We define incidence $I \subset \mathcal{P} \times \mathcal{L}$ as follows: $((p),[l]) \in I$ if and only if the following relations on their coordinates hold:

$$l_{11} - p_{11} = l_1 p_1, \qquad l_{12} - p_{12} = l_{11} p_1, \qquad l_{21} - p_{21} = l_1 p_{11},$$
$$l_{ii} - p_{ii} = l_1 p_{i-1,i}, \qquad l'_{ii} - p'_{ii} = l_{i,i-1} p_1, \qquad l_{i,i+1} - p_{i,i+1} = l_{ii} p_1,$$
$$l_{i+1,i} - p_{i+1,i} = l_1 p'_{ii}.$$

(The last four relations are defined for $i \geqslant 2$.)

Now for every integer $n \geqslant 2$ we obtain a geometry $(\mathcal{P}_n, \mathcal{L}_n, I_n)$ as follows: $\mathcal{P}_n$ and $\mathcal{L}_n$ are obtained from $\mathcal{P}$ and $\mathcal{L}$, respectively, by projecting each vector onto its n initial coordinates. Incidence I_n is then defined by imposing the first $n-1$ incidence relations and ignoring all others. For fixed q, the incidence graph corresponding to $(\mathcal{P}_n, \mathcal{L}_n, I_n)$ is denoted by $D(n,q)$. It is trivial to show that $D(n,q)$ is a q-regular bipartite graph on $2q^n$ vertices.

Graphs $D(n,q)$ were initially conceived in an attempt to generalize to graphs of larger girth the affine parts of generalized triangles and quadrangles. Thus, for q an arbitrary prime power, graph $D(2,q)$ is the incidence graph of the affine part of the generalized triangle of type $A_2(q)$, and $D(3,q)$ is the incidence graph of the affine part of the generalized quadrangle of type $B_2(q)$. Indeed we have used these graphs as our coordinatized models of the previous section.

Even with relatively simple models of graphs, identifying and describing polarities can be quite an arduous task. (Graph $D(3, 2^{2\alpha+1})$ from Section 3 provides an excellent example of this.) We present here a very natural and easily described polarity of graph $D(n,q)$ which exists whenever n is even and $q = 2^\alpha$, $\alpha \geqslant 1$. We define it by

$$(p)^\pi = [p_1, p_{11}, p_{21}, p_{12}, p'_{22}, p_{22}, p_{32}, p_{23}, \ldots, p'_{ii}, p_{ii}, p_{i+1,i}, p_{i,i+1}, \ldots],$$
$$[l]^\pi = (l_1, l_{11}, l_{21}, l_{12}, l'_{22}, l_{22}, l_{32}, l_{23}, \ldots, l'_{ii}, l_{ii}, l_{i+1,i}, l_{i,i+1}, \ldots).$$

In [19] it is proved that graphs $D(n,q)$ are disconnected for $n \geqslant 6$, and a complete description of the connected components is given in [20]. We remark here only that while, for fixed n and q, all components of $D(n,q)$ are isomorphic, not every component is stable under π when π in fact exists (n even, $q = 2^\alpha$). There is, however, always one π-stable component, namely the one which contains the zero point $(0) = (0,0,0,\ldots)$. We denote it by $\mathrm{CD}(n,q)$.

It is easy to see that there are precisely $q^{n/2}$ absolute points associated to π and these take the form

$$(0, a, b, b, c, c, d, d, \ldots).$$

Thus for n even and $q=2^{\alpha}$, graphs $\mathrm{CD}(n,q)$, and their polarity graphs $\mathrm{CD}(n,q)^{\pi}$, have the respective parameters:

$$v(\mathrm{CD}(n,q))\leqslant 2q^{s},\quad e(\mathrm{CD}(n,q))=q^{s+1},\quad g(\mathrm{CD}(n,q))\geqslant n+4,$$
$$v(\mathrm{CD}(n,q)^{\pi})=q^{s}\quad e(\mathrm{CD}(n,q)^{\pi})=q^{s+1}-q^{n/2},$$

where $s=n-\lfloor\frac{n+2}{4}\rfloor+1$. Graphs $\mathrm{CD}(n,q)$ therefore satisfy $e\geqslant(\frac{1}{2})^{1+\frac{1}{s}}v^{1+\frac{1}{s}}$, while their polarity graphs are $(n+2)$-cycle-free and satisfy $e\geqslant\frac{1}{2}v^{1+\frac{1}{s}}$.

Acknowledgements

We are grateful to Christoph Hundack for bringing to our attention Ref. [10]. We are grateful to an anonymous referee for several corrections, several references and many suggestions which improved the original version of this paper.

References

[1] C.T. Benson, Minimal regular graphs of girth eight and twelve, Can. J. Math. 18 (1966) 1091–1094.
[2] B. Bollobás, Extremal Graph Theory, Academic Press, London, 1978.
[3] J.A. Bondy, M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974) 87–105.
[4] A.E. Brouwer, The complement of a geometric hyperplane in a generalized polygon is usually connected, in: F. De Clerck et al. (Eds.), Finite Geometry and Combinatorics London Math. Soc. Lecture Note Ser. 191, Cambridge Univ. Press, Cambridge, 1993, pp. 53–57.
[5] W.G. Brown, On graphs that do not contain a Thomsen graph, Can. Math. Bull. 9(3) (1966) 281–285.
[6] R.W. Carter, Simple Groups of Lie Type, Wiley, New York, 1972.
[7] C.R.J. Claphan, A. Flockhart, J. Sheehan, Graphs without four-cycles, J. Graph Theory 13 (1989) 29–47.
[8] V. De Smet, H. Van Maldeghem, The finite Moufang hexagons coordinatized, Beiträge Algebra Geom. 34 (1993) 217–232.
[9] P. Erdős, A. Rényi, V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[10] P. Erdős, M. Simonovits, Compactness results in extremal graph theory, Combinatorica 2(3) (1982) 275–288.
[11] W. Feit, G. Higman, The non-existence of certain generalized polygons, J. Algebra 1 (1964) 114–131.
[12] R.J. Faudree, M. Simonovits, On a class of degenerate extremal graph problems, Combinatorica 3(1) (1983) 83–93.
[13] Z. Füredi, Graphs without quadrilaterals, J. Combin. Theory Ser. B 34 (1983) 187–190.
[14] Z. Füredi, Quadrilateral-free graphs with maximum number of edges, Preprint.
[15] G. Hanssens, H. Van Maldeghem, A new look at the classical generalized quadrangles, Ars Combin. 24 (1987) 199–210.
[16] F. Lazebnik, V.A. Ustimenko, New examples of graphs without small cycles and of large size, Eur. J. Combin. 14 (1993) 445–460.
[17] F. Lazebnik, V.A. Ustimenko, Explicit construction of graphs with an arbitrary large girth and of large size, Discrete Appl. Math. 60 (1995) 275–284.
[18] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, Properties of certain families of $2k$-cycle free graphs, J. Combin. Theory Ser. B 60(2) (1994) 293–298.
[19] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, A new series of dense graphs of large girth, Bull. Amer. Math. Soc. 32(1) (1995) 73–79.

[20] F. Lazebnik, V.A. Ustimenko, A.J. Woldar, A characterization of the components of the graphs $D(k,q)$, Discrete Math. 157 (1996) 271–283.
[21] L.P. Peykre, An automorphism of the geometry of the group $B_2(2^n)$ as a polynomial mapping, Ukrainian Math. J. 44(11) (1992) 1530–1534 (in Russian).
[22] J.A. Thas, Generalized polygons, in: F. Buekenhout (Ed.), Handbook on Incidence Geometry, North-Holland, Amsterdam, 1995, Ch. 9.
[23] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, Publ. Math. I.H.E.S. 2 (1959) 14–20.
[24] J. Tits, Les groupes simples de Suzuki et de Ree, Séminaire Bourbaki 13(210) (1960/1961) 1–18.
[25] M. Simonovits, Extremal graph theory, in: L.W. Beineke, R.J. Wilson (Eds.), Selected Topics in Graph Theory 2, Academic Press, London, 1983, pp. 161–200.
[26] V.A. Ustimenko, Linear interpretation of Chevalley group flag geometries, Ukr. Math. J. 43(7,8) (1991) 1055–1060 (in Russian).
[27] V.A. Ustimenko, On the varieties of parabolic subgroups, their generalizations and combinatorial applications, Acta Applicandae Mathematikae 52 (1998) 223–238.
[28] R. Wenger, Extremal graphs with no C^4, C^6, or C^{10}'s, J. Combin. Theory Ser. B 52 (1991) 113–116.